Chapter 8

Web Extension: The Binomial Approach

*The example in the chapter illustrated the **binomial approach.** This Web Extension explains the approach in more detail and illustrates its application when the year is divided into more than one period.*

The current stock of Western Cellular, P_0, sells for \$40 per share. Options exist that permit the holder to buy one share of Western at an exercise price of \$35. These options will expire at the end of one year. The steps to the binomial approach are shown below.

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*See the **FM11 Ch 08 Tool Kit.xls** for all the calculations.*

Step 1. Define the possible ending prices of the stock. Suppose our analysis of Western's stock indicates that the standard deviation (σ) of its expected annual stock return is 22.314 percent. The binomial approach assumes that Western's stock will be selling at one of two prices at the end of the period. In particular, it assumes the initial stock price will go up by a multiplicative factor (u) or go down by a multiplicative factor (d). In other words, the ending stock price for an upward movement, P_u, is $P_u = u(P_0)$, and the ending stock price for a downward movement, P_d, is $P_d = d(P_0)$. The trick is to choose u and d so that the resulting standard deviation of stock returns is equal to the desired annual standard deviation of 22.314 percent. The derivation is beyond the scope of a financial management textbook, but the appropriate equations are:

$$u = e^{\sigma\sqrt{T/n}} \tag{8E-1}$$

and

$$d = \frac{1}{u} \tag{8E-2}$$

where T is the time in years until the option expires and n is the number of steps per year.

Western's option has one year until it expires, and we are initially assuming there is only one step during the year. Therefore, u and d are

$$u = e^{\sigma\sqrt{T/n}} = e^{0.22314\sqrt{1/1}} = 1.25.$$

$$d = \frac{1}{u} = \frac{1}{1.25} = 0.80.$$

Based on these values of u and d, the ending stock prices are P_u = 1.25(\$40) = \$50 and P_d = 0.80(\$40) = \$32. Notice that if Western were a riskier stock, then its standard deviation would be higher than 22.314 percent. This would cause u to be higher than 1.25 and d to be lower than 0.80, resulting in a wider range of possible ending stock prices. Figure 8E-1 illustrates the stock's possible price paths when $\sigma = 22.314\%$; the other information in the figure is explained below.

STEP 2. Find the range of values at expiration. When the option expires at the end of the year, Western's stock will sell for either \$50 or \$32, a range of \$50 − \$32 = \$18. As shown in Figure 8-2 in the chapter, the option will pay \$15 if the stock is \$50, because this is above the exercise price of \$35: \$50 − \$35 = \$15. The option will pay nothing if the stock price is \$32, because this is below the exercise price. The range of option payoffs is \$15 − \$0 = \$15. The hedger's portfolio consists of the stock and the obligation to satisfy the option holder, so the value of the portfolio in one year is the stock price minus the option payoff.

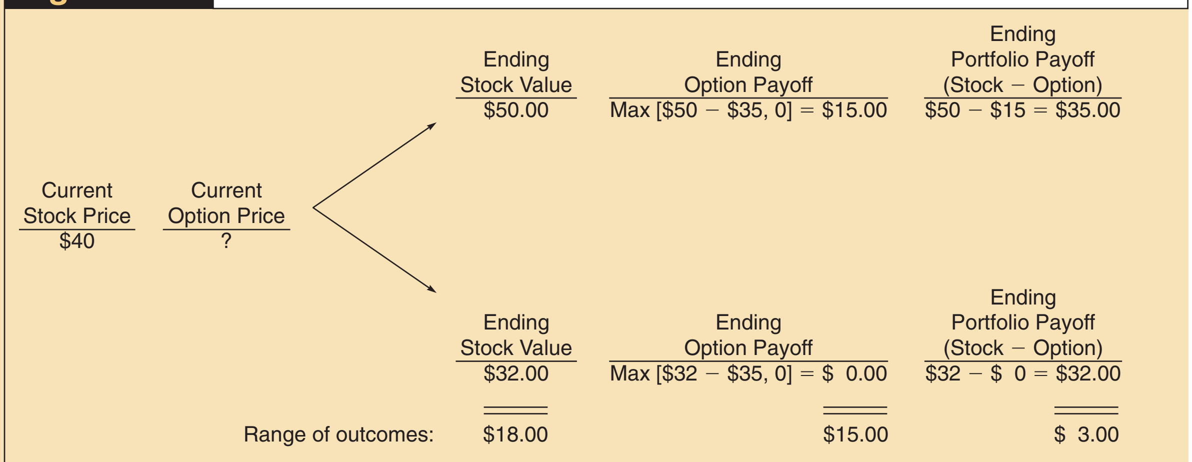
STEP 3. Buy exactly enough stock to equalize the range of payoffs for the stock and the option. Figure 8-2 shows that the ranges of payoffs for the stock and the option are \$18 and \$15. To construct the riskless portfolio, we need to equalize these ranges so that the profits from the stock exactly offset the losses in satisfying the option holder. We do so by buying \$15/\$18 = 0.8333 share and selling one option (or 8,333 shares and 10,000 options).

Here is why equalizing ranges gives the correct number of shares of stock. Let C_u be the call option payoff if the stock goes up, C_d the call option payoff if the stock goes down, and N the number of shares of stock. We want the portfolio value to be the same whether the stock is high or low. The portfolio value for a high stock price is $N(P_u) - C_u$, and the value for a low stock price is $N(P_d) - C_d$. Setting these equal and solving for N yields

$$N = \frac{C_u - C_d}{P_u - P_d},$$

which is the same as equalizing the ranges. Here is why equalizing ranges gives the correct number of stock shares.

Figure 8E-1 Binomial Approach



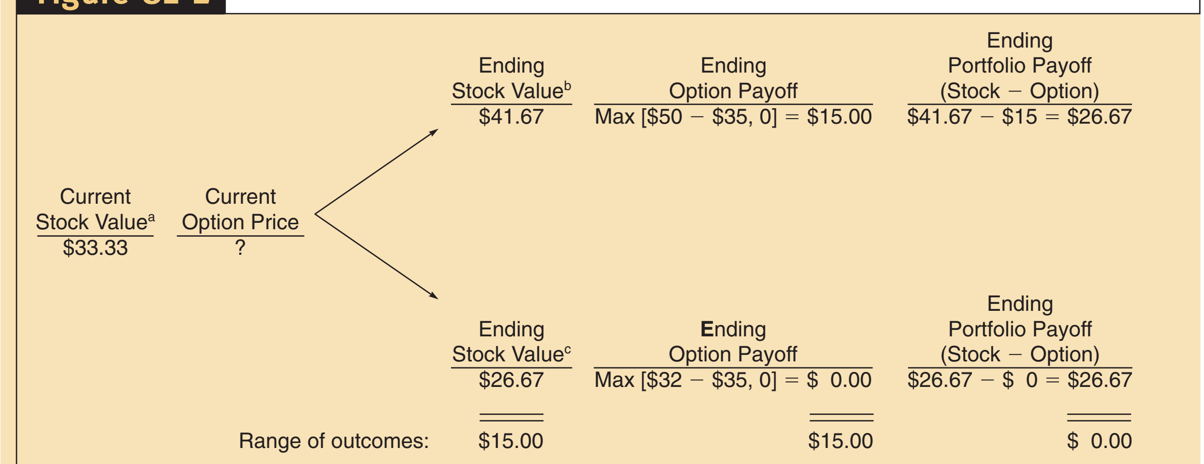
folio is \$40(0.8333) = \$33.33. The value of the portfolio's stock at the end of the year will be either \$50(0.8333) = \$41.67 or \$32(0.8333) = \$26.67. As shown in Figure 8E-2, the range of the stock's ending value is now \$41.67 − \$26.67 = \$15.

STEP 4. **Create a riskless hedged investment.** We created a riskless portfolio by buying 0.8333 share of the stock and selling one call option, as shown in Figure 8E-2. The stock in the portfolio will have a value of either \$41.67 or \$26.67, depending on the ending price of Western's stock. The call option that was sold will have no effect on the value of the portfolio if Western's price falls to \$32, because it will not be exercised—it will expire worthless. However, if the stock price ends at \$50, the holder of the option will exercise it, paying the \$35 exercise price for stock that would cost \$50 on the open market. The option holder's profit is the option writer's loss, so the option will cost the hedger \$15. Now note that the value of the portfolio is \$26.67 regardless of whether Western's stock goes up or down, so the portfolio is riskless. A hedge has been created that protects against both increases and decreases in the price of the stock.

STEP 5. **Find the call option's price.** To this point, we have not mentioned the price of the call option that was sold to create the riskless hedge. What is the *fair,* or *equilibrium,* price? The value of the portfolio will be \$26.67 at the end of the year, regardless of what happens to the price of the stock. This \$26.67 is riskless, and so the portfolio should earn the risk-free rate, which is 8 percent. In the chapter, we used daily compounding; technically, we should use continuous compounding. The present value of the portfolio's ending value is

$$PV = \$26.67e^{-r_{RF}t} = \$26.67e^{-0.08(1)} = \$24.62.$$

Figure 8E-2 The Hedge Portfolio



Notes:

[a]The portfolio contains 0.8333 share of stock, with a stock price of \$40, so its value is 0.8333(\$40) = \$33.33.

[b]The ending stock price is \$50, so the value is 0.8333(\$50) = \$41.67.

[c]The ending stock price is \$32, so the value is 0.8333(\$32) = \$26.67.

This means that the current value of the portfolio must be \$24.62 to ensure that the portfolio earns the risk-free rate of return. The current value of the portfolio is equal to the value of the stock minus the value of the obligation to cover the call option. At the time the call option is sold, the obligation's value is exactly equal to the price of the option. Because Western's stock is currently selling for \$40, and because the portfolio contains 0.8333 share, the value of the stock in the portfolio is 0.8333(\$40) = \$33.33. What remains is the price of the option:

$$\begin{aligned} \text{PV of portfolio} &= \text{Current value of stock in portfolio} - \text{Current option price} \\ \text{Current option price} &= \text{Current value of stock in portfolio} - \text{PV of portfolio} \\ &= \$33.33 - \$24.62 = \$8.71. \end{aligned}$$

If this option sold at a price higher than \$8.71, other investors could create riskless portfolios as described above and earn more than the riskless rate. Investors (especially the large investment banking firms) would create such portfolios and sell options until their price fell to \$8.71, at which point the market would be in equilibrium. Conversely, if the options sold for less than \$8.71, investors would create an "opposite" portfolio by buying a call option and selling short the stock. The resulting supply shortage would drive the price up to \$8.71. Thus, investors (or arbitrageurs) would buy and sell in the market until the options were priced at their equilibrium level.

Clearly, this example is unrealistic. Although you could duplicate the purchase of 0.8333 shares by buying 8,333 shares and selling 10,000 options, the stock price assumptions are unrealistic; Western's stock price could be almost anything after one year, not just \$50 or \$32. However, if we allowed the stock to move up or down more often during the year, then a more realistic range of ending prices would result. In fact, if we allowed hundreds, or even thousands, of up and down stock price movements during the year, the resulting distribution of stock prices would approximate the distributions of actual stocks. To see how the binomial approach can accommodate more than one up or down movement in the year, suppose we allowed the stock to move up or down every six months. As before, we must choose u and d so that the resulting standard deviation of annual stock returns is equal to the desired standard deviation of 22.314 percent. Western's option still has one year until it expires, but we are now assuming there are two steps during the year. Therefore, the new values of u and d are

$$u = e^{\sigma\sqrt{T/n}} = e^{0.22314\sqrt{1/2}} = 1.1709.$$

$$d = \frac{1}{u} = \frac{1}{1.1709} = 0.8540.$$

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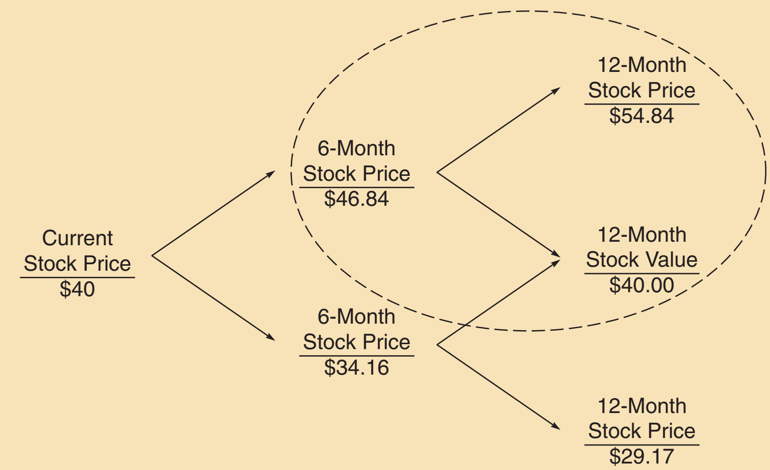
*See **FM11 Ch 08 Tool Kit.xls** for all calculations.*

Based on these values of u and d, the ending stock price for two upward movements is 1.1709(1.1709)(\$40) = \$54.84. For two downward movements, the ending stock price is 0.8540(0.8540)(\$40) = \$29.17. If the stock goes up and then down, the ending price is 1.1709(0.8540)(\$40) = \$40. The range of outcomes is a little larger than in Figure 8E-1, but the standard deviation of stock returns is the same, because most of the time the stock is expected to end up in the middle, rather than on the top or bottom. Keeping the standard deviation the same as in Figure 8E-1 allows us to compare apples and apples, rather than apples and oranges. The pattern of stock prices shown in Figure 8E-3 is called a **binomial lattice**.

If we focus only on the portion of the lattice shown inside the oval, it is very similar to the problem we just solved in Figure 8E-1, except it has slightly different

Figure 8E-3 The Binomial Lattice



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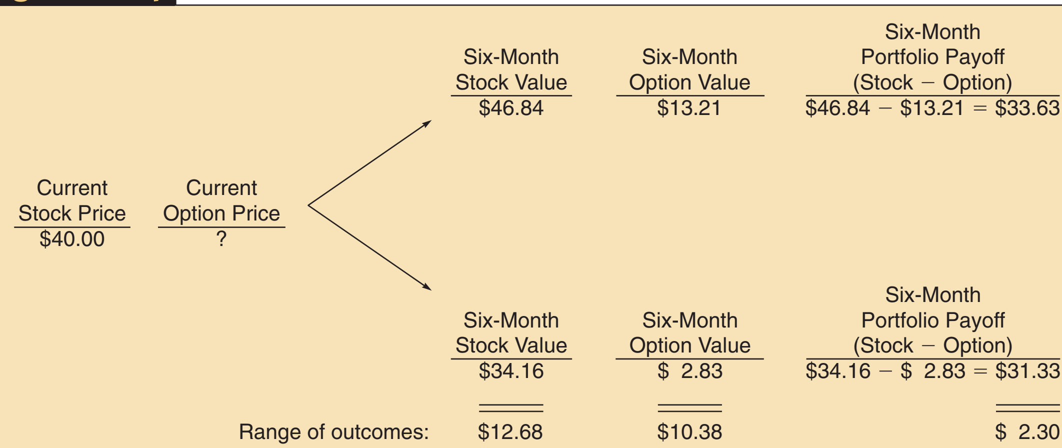
stock prices. We can apply the same solution procedure to find the price of the option at the end of six months, given a six-month stock price of $46.84. As shown in ***FM11 Ch 08 Tool Kit.xls,*** the value of the option at the end of six months is $13.21, given that the stock price goes up to $46.84. Applying the same approach to the lower-right portion of the lattice, the ***Tool Kit*** shows that the option price at the end of six months is $2.83, given a six-month stock price of $34.16.

Figure 8E-4 shows the stock prices, the option values we just calculated, and the portfolio payoffs for the first six months. Using the same approach as before, we can create a hedge portfolio consisting of 0.8187 shares of stock and one short call option, as shown in the ***Tool Kit.*** This portfolio has a constant value at six months, whether the stock goes up or down. Just as before, we can find the present value of the portfolio, and then find the option price by subtracting the portfolios' present

Figure 8E-4 Stock, Option, and Portfolio Values in Six Months



There are many free binomial option-pricing programs on the Web, including one at ***http://www.kellogg.nwu.edu/faculty/benzoni/ftp/D6o/Files/options/options.xls.***

value from the value of the stock in the portfolio. As the ***Tool Kit*** shows, the current price of the option is $8.60. Notice that this is a little lower than the $8.71 price we estimated earlier with only one period in the year.

If we break the year into smaller periods and allow the stock price to move up or down more often, the lattice would have a more realistic range of possible outcomes. Of course, estimating the current option price would require solving lots of problems within the lattice, but each problem is very simple, and computers can solve them very rapidly. With more outcomes, the resulting estimated option price is more accurate. For example, if we divide the year into 10 periods, the estimated price is $8.38. With 100 periods, the price is $8.41. With 1,000, it is still $8.41, which shows that the solution converges to its final value with a relatively small number of steps. In fact, as we break the year into smaller and smaller periods, the solution for the binomial approach converges to the Black-Scholes solution.

The binomial approach is widely used to value options with more complicated payoffs than the call option in our example. This is beyond the scope of a financial management textbook, but if you are interested in learning more about the binomial approach, take a look at the textbooks listed in the end-of-chapter references.